AROUND THE CUSP SINGULARITY AND THE BREAKING OF WAVES

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Abstract

WAVES is an "Art-Science" project on water surface waves. We aim to visualize the behaviour of water waves during their evolution: generation, focusing and breaking. Relying on the general property of waves to focus when properly generated or reflected, we use a parabolicaly shaped wave maker to focus water waves in a region of the water surface called the Huygens cusp in optics. We record these breakings using a fast video camera. A novel and spectacular vision of wave breakings obtained when playing at slow speed was presented during the workshop "Water is in the Air" in Marseilles, June 2012.

Light focusing

Descartes's laws of light wave reflexion is the fundamental law of geometrical optics. We are used to images through lenses or mirrors and to the focusing of light rays in some narrow bright regions of space. A simple glass of water enlighted by the sun rays may produce amazing light patterns with the appearance of complex shapes with sharp edges. These lines where the light rays are focused are called caustics as they are associated to quite high temperature elevations. Focusing light at the bottom of a tea cup is another exemple of every day experience, but a precise observation of the light pattern shows a very simple but singular bright curve with a clearly visible cusp. Figure 1 shows two pictures of this "cusp in the cup" which further than its beautiful and intriguing simplicity hides as we will see later, some amazing geometrical properties.

The mathematical descriptions of these types of singular curves originates from the classification of "Catastrophes" by René Thom [1] and later by Erik Zeeman [2].

Catastrophe theory permits the analysis of the shape of functions depending on some parameters.

Fig. 1. A cusp in a cup illuminated by sun rays.

It permits to detect singular points where the derivatives of the considered function become singular. In a neighborough of these points, the function can then be approximated by its polynomial Taylor expansion. The order of these expansions and the number of the free parameters determine finally the classification of Elementary Catastrophes as described by Thom[1]. The cusp (*la fronce*, in French) is the simplest catastrophe depending on two parameters. The drawing given in figure 2 illustrates this three dimensionnal folding of a surface whose equation is given by $\mathbf{Z^3 - \mathit{a}\, Z - \mathit{b} = 0}$, where a and b are the free parameters of the model. As can be seen, in a certain region of the

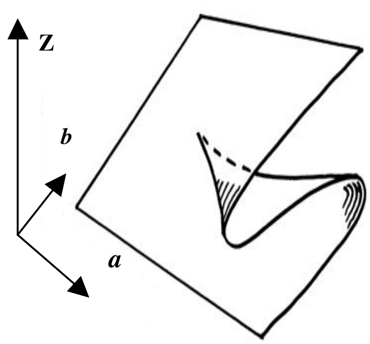


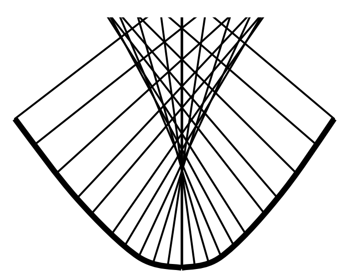
Fig.2. Drawing of the cusp singularity of the folded manifold $\mathbf{Z^3 - \mathit{a}\, Z - \mathit{b} = 0}$

parameter space (a,b), Z takes a single value, whereas in another domain which is delimited inside the curve defined by zeroing the discriminant of the cubic equation: $\mathbf{4\,\mathit{a}^3 - 27\,\mathit{b}^2 = 0}$, the equation for Z can have three solutions. The deliminating curve $\mathbf{4\,\mathit{a}^3 - 27\,\mathit{b}^2 = 0}$, is called the Huygens cusp which is the geometrical interpretation of the light ray focusing as it was observed in the cup (see figure 1). Indeed, in the bright region, three light rays converge after refexion on the cup wall. On the contrary, at any point outside the cusp, a single ray is only found.

Water waves make waves visible

As a matter of fact, and even if we extensively use them to communicate, light waves and sound waves stay more or less invisible to us. We can see or hear a signal or an image but during their transmission or their propagation through air, water or even vaccum for light, waves stay completely invisible to us. This is so true that in a physics class on waves, water waves are commonly used to illustrate the basic laws of wave propagation, reflexion or diffraction. Therefore, the idea of producing caustics and cusps when focusing hydrodynamical waves seems natural. Amazingly, it is only quite recently that this analogy between light waves and hydrodynamic waves was pointed out in the general context of pattern formation [3]. Later, She et al. [4] were the firsts to explicitely prepare a wave field where the wave energy is focused by an appropriate choice of the phase lag between the 75 paddles they used to generate the waves. More recently, thanks to the increased power of super computers, the full three dimensional numerical simulation of wave focusing has been carried out in order to simulate freak wave generation [5].

Fig.3. Rays generated by a parabolic emitter focus in the Huygens cusp



Following the ideas of She et al. [4], we designed a parabolic wave maker in order to generate waves that will focus in the Huygens cusp associated to the parabola. Figure 3 presents the ray pattern associated to waves propagating away from the parabola and that draws the cusp. Geometrical optics ray theory predicts a divergence of the wave energy at the cusp. In fact, it is well known that at small scale (here the wavelength of the wave, its color for light) geometrical singularities are softened by diffraction. The full calculation of wave amplitude in the neighborough of the cusp that uses diffraction theory, was first accomplished by Pearcey [6]. This theory was then recently extended to tsunami waves focused by ocean bottom lenses [7] that enhance the amplitude and the power of these destructive waves.

The breaking of water waves

Several types of wave breaking have been recognized by oceanographers. As an example, it is known that non linear shallow water waves break because of the appearance of a singularity in the shape of the water surface forming a vertical front just before the breaking. Taking the advantage of this singular behavior, general scaling laws have been deduced [8]. These universal laws originate from the geometrical shape of the cusp as seen before. Imagine that the surface Z represented in figure 2 is the water surface where the parameter a would be the time t and b the spatial variable perpendicuar to the water crest. The folded manifold of figure 2 is seen now as a spatio-temporal representation of a breaking wave. As shown before, the breaking can be interpreted as the lost of uniqueness of the coordinate Z of the water surface as a function of b. From the simple geometrical relation deduced before and linking parameters a and b , we can easily deduce that the length of the water tongue plunging ahead of the crest should scale as $t^{3/2}$. This scaling law was first checked numerically in [9] and we will present here for the first time its experimental verification.

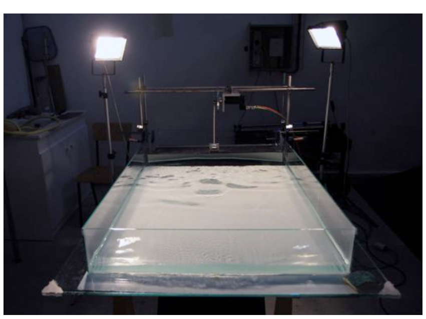


Fig.4. The water table with its wave-maker and illuminating equipment.

With this intention, we have installed a water table (1.4 m long and 1 m wide) that can be seen on figure 4. The table is equiped with a parabolic wave maker whose motion is driven by a servo-motor. The depth of the water layer is chosen around 3 cm. The progression of a solitary wave generated at one end of the table and focusing at a cusp located 1 meter ahead of the wave emitter, is recorded by a fast video camera at a rate of 2000 images per second. Figure 5 shows two snapshots extracted from one the movies during the breaking. The first image shows the very early time of the water tongue plunging ahead of the wave crest. Soon after, some capilarity wavelets are clearly visible.

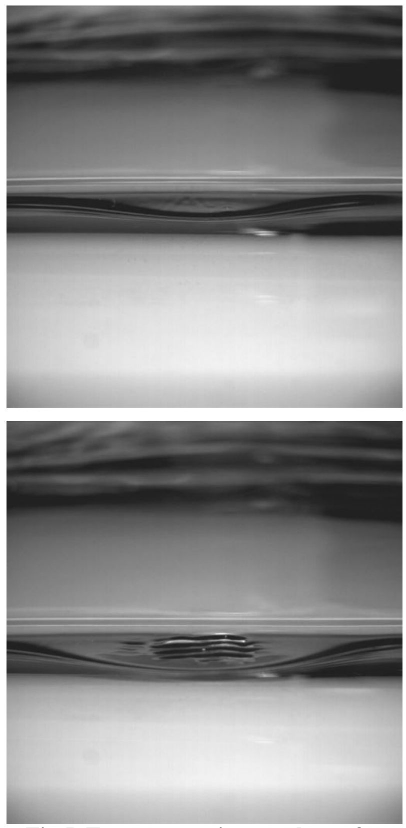
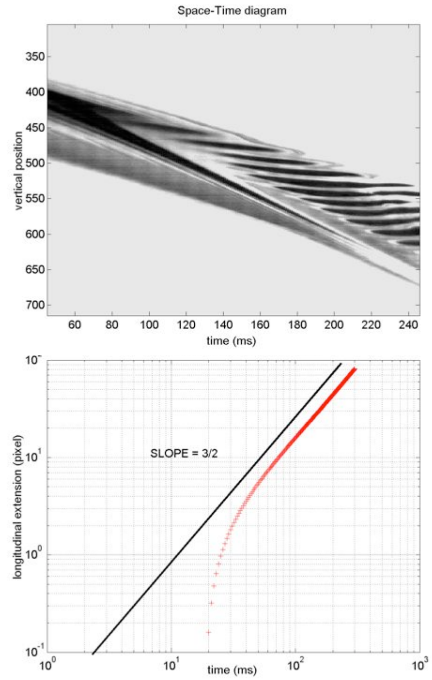


Fig 5. Two consecutive snaphots of the breaking caused by the focusing of a solitary wave in a cusp.

From the movies, we can construct space-time diagrams by extracting vertical video lines from each images and piling them together. Figure 6 illustrates this dynamics of the breaking and in particular the appearance of the cusp with its amazing stationnary capilllarity wave decoration. The image analysis of these space time data shows the measurement of the expected 3/2 power of time as dictated by the cusp geometry. To our knowledge this is the first time that this law is observed from experiments.

Fig.6. Space-time diagram extracted from the video movie of the breaking and logarithmic plot of the border lines of the cusp showing the $t^{3/2}$ behavior.



The artist point of view (Javiera Tejerina-Risso[1,2])

As a non-classical collaboration between two persons from different disciplines, in this case, a scientist and an artist, it seems logical to continue the confrontation of two different points of view: I give in this companion paper my point of view about this collaboration during my residency at IMéRA to develop this project.

Into the right tempo

Taking part into a residency program, in a specific place, a company or an institution, means above all, a time of discovery and a meeting between two different space-time: the one of the artist and the one of the hosting institution. Taking part into a residency program in a science laboratory means a deep immersion into a universe I barely know, with its own codes and practices. Creating an art-science project is exactly the point of convergence of two different worlds that share the same fascination of observing the world around us.

To be able to create such a convergence, you must have a shared object, a middle ground. Between Patrice Le Gal, the researcher and me, the artist, there is a common fascination for flows. He studies, reproduces, analyses, creates models of the flow phenomenon and then, he spreads his research among his peers. For me, the flows are raw material that I can use to create my video pieces, installation and photographs thanks to their physical features, their plasticity but also their evocative meaning of time passing by and memories.

Our research was about finding this common object: the one the scientist could use for his data and the one I could appropriate for its shape and the story it tells.

The sea has a central place in my work because it is related to the concept of traveling, exile, new horizons and the existence of an elsewhere. Under different forms, I have used this element: the sea. Using its flows and ebbs. Figure 7 shows two snapshots of my video work "Blanc/Bleu" where the Mediterranean Sea becomes the support of drawings made by light reflections and the sea movement. The water, multiple by essence, is also a symbol of impermanence, it is both eternal and in constant change. This is the paradox I want to grasp with my work. Figure 8 shows two pictures of the video installation I did in 2012. It establishes a dialogue between three movies about the water, a river, a sea and a lake, each of them with its own temporality and power.

I could never forget the sound of Pacific Ocean waves breaking against the rocks in the Chilean cost. Those powerful waves remain in my memories…

When I arrived to the IRPHE lab, Patrice Le Gal had already developed a research about the experimental breaking of the wave. Once again we were converging our interests into this common object, our wave project.

Fig 7. Blanc/Bleu, Javiera Tejerina-Risso, 2010, two stills from the video

It seemed then completely natural to continue to develop this path: looking for the experimental focusing of a wave in shallow water. It was interesting for me to recreate a single wave in a water tank that we could shape as we wanted in order to give its owns characteristics. From the powerful ocean wave we came to a smooth, fragile and delicate shape: our wave.

Into the water tank, from a still water, the wave shapes, advances, stands, it becomes a wave. Then, in a fleeting period of time, it breaks into a sublime beauty to disappear again. It loses its uniqueness. It melts into the whole and so on.

Everything is a matter of time

The geometry of the space, the mutant shape of the wave is therefore synonym of the time. In order to grasp the whole transformation of the wave, it was necessary to use a high-speed camera. Thanks to this technology we could grasp all the different shapes the wave wears while it is disappearing, into the breaking wave.

This was what I wanted to tame, what is always running, the escaping shape.

We ran different tests trying to get the perfect wave, symmetrical, converging in the exact point calculated by geometrical laws. And then, we got it. It was, what we called, our magic moment. Beyond our expectations, we get what we were both looking for: the perfectly–shaped wave that we could use both in a scientific and artistic way, or even better, for our art-science project. It is hard to explain, but there was this magic moment, were we both agreed we had our common object. Each of us knew why it was perfect for each one of us. Somehow we also knew that the other one was also satisfied with the results.

Now what it is interesting for me is to manipulate the waves images taken in the lab by playing with shrinking or expanding the time. Therefore I could get a huge range of different wave shapes, its metamorphoses. Then I could confront each of this temporality, each picture in a common dialogue.

Films and stills can attest of this stealthy moment, where the wave grows and dies.

Concluding remarks

Although the breaking of water waves is quite usual along sea shores, the physical mechanisms that generate this singular tilt-over of the wave crests are not yet fully understood.

Using an analogy between water waves and light, a joint research on wave focusing, which combines both an artistic approach and a scientific requirement allowed us to obtain unprecedented images of breaking waves. These observations lead to the measure of a scaling exponent of the breaking as given by the geometry of the underlying cusp as described by the catastrophe theory.

These experience allow us, as well, to create a forthcoming serie of artworks: video installations and photographs.

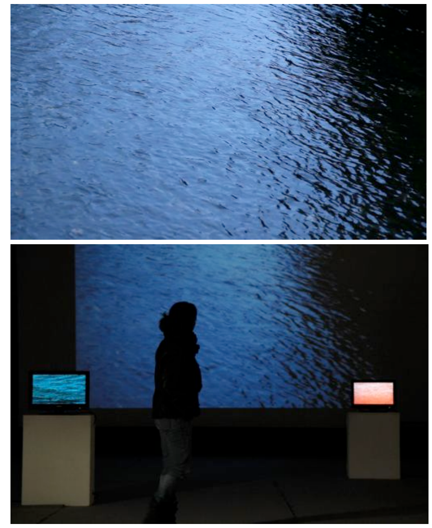


Fig 8. Video installation by Javiera Tejerina-Risso featuring the bichromies serie : Blanc/Bleu, Blanc/Rouge & Noir/Bleu in 2012.

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